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Growing Classical and Quantum Entropies in the Early Universe

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Abstract Following the idea that the global and local arrow of time has a cosmological origin, we define an entropy in the classical and in the quantum periods of the universe evolution. For the quantum period a semi-classical approach is adopted, modelling the universe with Wheeler-De Witt equation and using WKB. By applying the self-induced decoherence to the state of the universe it is proved that the quantum universe becomes a classical one. This allows us to define a conditional entropy which, in our simplified model, is proportional to $e^{2\gamma t}$ where γ is the dumping factor associated with the interaction potential of the scalar fields. Finally we find both Gibbs and thermodynamical entropy of the universe based in the conditional entropy.

Keywords Entropy · Quantum universe · Decoherence

1 Introduction

In the scientific literature the concept of entropy in a cosmological context is not yet completely understood. The purpose of this paper is to give an account of a possible link between quantum processes occurring in the initial phase of the universe and the Gibbs entropy, and to frame the obtained founded results within our theory of the arrow of time. For this purpose we have combined several mathematical formalism that it will be presented in short but, we hope, comprehensive version. Presently we do not see a way to simplify this procedure but in order to keep things as simple as possible we have used the simplest cosmological model. In this introduction we briefly review the previous results on this subject.

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1.1 The Arrow of Time

As it well known, the notion of an ever growing entropy is not directly based on the equations of fundamental physics, since they are all time-reversal invariant and the corresponding dynamical evolutions are unitary. As a consequence, any future directed irreversible-like phenomenon has a symmetric phenomenon directed to the past direction. We have called these pairs "time-symmetric twins" [2]. In fact, since 1912 [36] it is known that for any growing entropy evolution we can find its time-symmetric twin: a decaying entropy evolution. Then, in order to correctly define an entropy, some additional ingredients have to be introduced. This is the reason why we have not used entropy to define the arrow of time in our previous works [21, 23, 30, 31, 33]. On the contrary, we have based the arrow of time on the cosmological asymmetry of the universe. Precisely, the arrow of time is defined by a cosmological and substantial (non conventional) difference between past and future [57]. At the cosmological level, this difference is the time-asymmetry of the generic universe (like ours), as we have proved in [31]. The global arrow can be transferred to local contexts using the energy-momentum tensor, which also turns out to be time-asymmetric (in the case where the dominant energy condition is satisfied). For a complete understanding of this line of research it is recommended to read references [2] and [31]. In this way, all the usual arrows of time can be recovered (electromagnetic, quantum, intuitive, etc.). We have even found the irreversible thermodynamic arrow of time in [2], but other thermodynamic arrows of time are still missing from our proposal. In this paper we will introduce in this scheme the quantum and classical Gibbs and Conditional entropies in the early quantum universe and in its semiclassical limit in some particular cosmological models. For this purpose we must use the notion of decoherence.

1.2 Quantum Decoherence

Decoherence in open systems is explained by the EID in papers [55, 63–70]. But this approach has at least three problems:

- Most important, it cannot be applied to closed systems as the universe. Then the universe must be, more or less arbitrarily, divided in two parts, the proper system and the environment (as in papers [52] and [5]).
- EID does not provide a definition of proper system and environment. W. Zurek call this fact "the looming big problem of EID" [71]. This problem is really critical when the system is the universe, where all its parts must be considered as essentially equivalent.
- The pointer basis is not well defined (see [26, 29]).

For all this reasons and being the universe a closed system, several authors have introduced decoherence formalism for closed system [1, 9–11, 34, 35, 38, 53, 56, 59]. Three important examples are given in [37] where it is shown a system that decoheres at high temperature, in paper [12], where a Sinai-Young model is presented, where complexity produces decoherence in a closed triangular box, and in [39] where the decoherence appears in a closed system, i.e. the universe, if real clock are used. Also we have developed our own theory for decoherence of closed systems, SID (see [13, 15, 19, 24, 25, 27]) in the same basis of the one of paper [39]. In paper [14] we show how our formalism explain the decoherence of the Sinai-Young model above.

In the Appendix we give a short account of SID with concepts relevant for the paper.

1.3 Decoherence in the Quantum Universe

In cosmological models, entropy can also be computed introducing a coarse-graining on the effective action of the theory. E.g. in the context of Environment Induced Decoherence (EID) Lombardo and Mazzitelli [52] define this coarse-graining considering a field of matter, and integrating all modes of this field with wavelength smaller than a critical value. Then an effective action is obtained, i.e. the equation of evolution for the reduced density matrix. Calculating the diffusion coefficients of this equation induced decoherence can be analyzed through the modes of wave length larger than the critical value. These results are applied to the case of a de Sitter space-time interacting with a coupled scalar field. It is shown that decoherence is effective as long as the critical wavelength is not shorter than the Hubble radius. There is an extensive literature on decoherence in quantum cosmology which, at the level of our present knowledge about quantum gravity, has led to a consistent picture of quantum to classical transition in cosmology (see [5, 44–48, 62]). Nevertheless in this paper we will follow a different approach, using a set of three global coarse graining, we will find an evergrowing entropy of the universe in the quantum period in order to complete this panorama.

1.4 Line of Thought

Our line of thought will be the following. In Sect. 2.1 we will choose, as our model, the Robertson-Walker flat universe, and its quantum version. In Sect. 2.2 we will explain its decoherence process according to SID. Then in Sect. 2.2.3 we will introduce three successive reduction of the space of observables (all of them are quite usual in the literature). Each one of these reduction can be considered as a "coarse graining". In this way we will introduce: *reduction I*, namely to only use the van Hove observables, then *reduction II,* namely to use just analytical functions, and finally *reduction III* where we will only retain the exponential evolution of the universe. In Sect. 3 we will define an evergrowing conditional entropy and in Sect. 4 the corresponding Gibbs and thermodynamical entropies. Then in Sect. 5 we will draw our conclusions. An Appendix, reviewing the SID approach, completes the work.

2 Decoherence in a Closed Universe

If the transition from quantum to classical does not require the split of the universe into subsystems as a necessary condition, then decoherence must be one of the processes that explain how the universe as a whole becomes classical and reaches the thermic equilibrium state. In this section we will apply the self-induced approach SID to a simple semi classical-cosmological model in order to show how classically arises in this case.

2.1 The Model

Let us consider the flat Robertson-Walker universe with a metric:

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{1}$$

where η is the conformal time and a the scale of the universe. Let us consider a free neutral scalar field Φ and let us couple this field with the metric, with a conformal coupling ($\xi = \frac{1}{6}$). The total action reads $S = S_g + S_f + S_i$, and the gravitational action is:

$$S_g = M^2 \int d\eta \left[-\frac{1}{2}\dot{a}^2 - V(a) \right] \tag{2}$$

where M is the Planck mass, $\dot{a} = da/d\eta$, and the potential V contains the cosmological constant term and, eventually, the contribution of some form of classical matter. We suppose that V is such that the universe becomes asymptotic in equilibrium for a radius of the order of a_1. We expand the field Φ as:

$$\Phi(\eta, \vec{x}) = \int_{-\infty}^{+\infty} f_{\vec{k}}(\eta) e^{-i\vec{k}\cdot\vec{x}} d\vec{k} \tag{3}$$

where the components of $\vec{k} \in R^3$ are three continuous variables. The Wheeler-De Witt equation for this model reads:

$$H\Psi(a, \Phi) = (h_g + h_f + h_i)\Psi(a, \Phi) = 0 \tag{4}$$

where:

$$h_g = \frac{1}{2M^2}\partial_a^2 + M^2 V(a), \tag{5}$$

$$h_f = -\frac{1}{2}\int \left(\partial_{\vec{k}}^2 - k^2 f_{\vec{k}}^2\right) d\vec{k}, \tag{6}$$

$$h_i = \frac{1}{2} m^2 a^2 \int f_{\vec{k}}^2 d\vec{k} \tag{7}$$

with m the mass of the scalar field, $\vec{k}/a$ the linear momentum of the field, and $\partial_{\vec{k}} = \frac{\partial}{\partial f_{\vec{k}}}$. We can now go to the semiclassical regime using the WKB method ([40]), writing $\Psi(a, \Phi)$ as:

$$\Psi(a, \Phi) = \exp[iM^2 S(a)]\chi(a, \Phi) \tag{8}$$

and expanding S and χ as:

$$S = S_o + M^{-1} S_1 + \cdots \qquad \chi = \chi_o + M^{-1}\chi_1 + \cdots \tag{9}$$

To satisfy (4) at the order M^2, the principal Jacobi function $S(a)$ must satisfy the Hamilton-Jacobi equation:

$$\left(\frac{dS}{da}\right)^2 = 2V(a) \tag{10}$$

We can now define the (semi) classical time[1]

$$\frac{d}{d\eta} = \frac{dS}{da}\frac{d}{da} = \pm\sqrt{2V(a)}\frac{d}{da} \tag{11}$$

The solution of this equation is $a = \pm F(\eta, C)$ where C is an arbitrary integration constant. Different values of this constant and of the $\pm$ sign give different classical solutions for the geometry. Then, in the next order of the WKB expansion, χ satisfies a Schroedinger equation that reads:

$$i\frac{d\chi}{d\eta} = h(\eta)\chi \tag{12}$$

[1] We will consider the conformal time η of (11) defined through the *classical time* as $d\eta = \frac{dt}{a(t)}$.

where

$$h(\eta) = h_f + h_i(\eta) \tag{13}$$

precisely:

$$h(\eta) = -\frac{1}{2}\int \left[-\frac{\partial^2}{\partial f_{\vec{k}}^2} + \Omega_{\vec{k}}^2(a) f_{\vec{k}}^2 \right] d\vec{k} \tag{14}$$

where

$$\Omega_{\vec{k}}^2(a) = \Omega_{\omega}^2(a) = m^2a^2 + k^2 = m^2a^2 + \omega \tag{15}$$

where $\omega = k^2$. So the time dependence of the Hamiltonian comes from the function $a = a(\eta)$. Let us now consider a scale of the universe such that $a = a_{out} \gg a_1$. In this region the geometry is almost constant. Therefore, we have an adiabatic final vacuum $|0\rangle$ and adiabatic creation and annihilation operators $a_{\vec{k}}^{\dagger}$ and $a_{\vec{k}}$.[2] Then $h = h(a_{out})$ reads:

$$h = \int \Omega_{\omega} a_{\vec{k}}^{\dagger} a_{\vec{k}} d\vec{k} \tag{16}$$

We can now consider the Fock space and a basis of vectors:

$$|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle \cong |\{\vec{k}\}\rangle = a_{\vec{k}_1}^{\dagger} a_{\vec{k}_2}^{\dagger} \cdots a_{\vec{k}_n}^{\dagger} |0\rangle \tag{17}$$

where we have called $\{\vec{k}\}$ the set $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n$, where eventually n goes to infinity. The vectors of this basis are eigenvectors of h:

$$h|\{\vec{k}\}\rangle = \omega|\{\vec{k}\}\rangle \tag{18}$$

where

$$\omega = \sum_{\vec{k}\in\{\vec{k}\}} \Omega_{\omega} = \sum_{\vec{k}\in\{\vec{k}\}} (m^2a^2 + \omega)^{1/2} \tag{19}$$

For the sake of simplicity and to maintain the consistency with the notation of this paper, we can only use the energy to label the eigenvectors as:

$$|\{\vec{k}\}\rangle = |\Psi_{\omega}\rangle \tag{20}$$

where we have omitted the other components of the label $[\vec{k}]$ which really are necessary to unambiguously define the vector, as we will see below. Finally in this notation, the Hamiltonian reads:

$$h = \int \omega |\Psi_{\omega}\rangle\langle\Psi_{\omega}| d\omega \tag{21}$$

[2]The adiabatic vacuum is an approximation, for an expanding universe with a very low expansion rate, of the usual vacuum. With height expanding real the notion of vacuum is meaningless. This vacuum was introduce by L. Parker in [54] based in an idea of Einstein. Since then it is consider as the all most only reliable definition of vacuum and also the base of the notion of particle number in an expanding universe [7].

2.2 The Three Reductions

In order to obtain the entropy of the wave function of the universe there will see that three reductions in the space of observables are necessary: the first one will lead us to decoherence, the second and third will show the decaying evolution of this wave function. Once we have made this three reductions we will use the statistical quantum mechanics to define the entropy of the system, which in this case is the universe.

2.2.1 Decoherence and Equilibrium in Energy Basis (Reduction I)

We introduce the first particular choice or *reduction I*. In this case, a generic observable $|O) \in V_O^{VH}$ reads (cf. (70) of the Appendix) where V_O^{VH} is the van Hove space of observables (see Appendix):

$$|O) = \int O(\omega)|\Psi_\omega)d\omega + \int\int O(\omega,\omega')|\Psi_\omega,\Psi_{\omega'})d\omega d\omega' \tag{22}$$

where:

$$|\Psi_\omega) = |\Psi_\omega\rangle\langle\Psi_\omega| \tag{23}$$

$$|\Psi_\omega,\Psi_{\omega'}) = |\Psi_\omega\rangle\langle\Psi_{\omega'}| \tag{24}$$

and a generic state $(\rho| \in V_S^{VH}$ can be expressed as (cf. (71) of the Appendix):

$$(\rho| = \int \rho(\omega)(\Psi_\omega|d\omega + \int\int \rho(\omega,\omega')(\Psi_\omega,\Psi_{\omega'}|d\omega d\omega' \tag{25}$$

where $\{(\Psi_\omega|, (\Psi_{\omega'}|\}$, is the cobasis of $\{|\Psi_\omega), |\Psi_{\omega'})\}$. Then, the mean value of the observable $|O)$ in the state $(\rho(t)|$ reads (cf. (74) of the Appendix):

$$\langle O\rangle_{\rho(t)} = (\rho(t)|O) = \int \rho(\omega)O(\omega)d\omega + \int\int \rho(\omega,\omega')O(\omega,\omega')e^{-i(\omega-\omega')t}d\omega d\omega' \tag{26}$$

Taking the limit for $t \to \infty$ and applying the Riemann-Lebesgue theorem, we obtain (cf. (75) of the Appendix):

$$\langle O\rangle_{\rho(t)} = \lim_{t\to\infty}(\rho(t)|O) = \int \rho(\omega)O(\omega)d\omega \tag{27}$$

And this integral is equivalent to the mean value of the observable $|O)$ in a new state $(\rho_*|$:

$$(\rho_*| = \int \rho(\omega)(\Psi_\omega|d\omega \tag{28}$$

This new state $(\rho_*|$ is the equilibrium time-asymptotic state of the universe, which is diagonal in the variables ω, ω' as decoherence in energy requires, in this way, quantum universe asymptotically approaches to this state.

2.2.2 Exponential Evolution of the Wave Function of the Universe (Reduction II)

A problem related with irreversibility is originated on the fact that the usually observed decaying evolutions towards equilibrium are exponential during a long period of time, whereas quantum mechanics, even when the state is traced, precludes this kind of evolutions. Precisely there is a non exponential effect for short ties (Zeno effect) and another one for long times (Khalfin effect). Nevertheless in many cases it is usual to just consider the exponential law eliminating the so called Zeno and Khalfin components. Following our algebraic approach, we will prove that exponential evolutions can be obtained when we further reduce our spaces of observables and states as follows:

$$V_{VH}^{+} = V_{VH}^{S} \oplus V_{VH}^{R+} \qquad V_{VH}^{+'} = V_{VH}^{S'} \oplus V_{VH}^{R+'} \tag{29}$$

where $V_{VH}^{R+} \subset V_{VH}^{R}$ and $V_{VH}^{R+'} \subset V_{VH}^{R'}$ are subspaces endowed with particular analytical properties in the complex energy plane that we will precise below. We will call this second step *reduction II* or *analytical reduction*. Let us consider the following spaces:

$$V_{VH}^{R+} = \Phi_{+} \otimes \Phi_{+} \qquad V_{VH}^{R+'} = \Phi_{-} \otimes \Phi_{-}$$

where $|\Psi\rangle \in \Phi_{+}$ if, being $|\Psi_{\omega}^{f}\rangle$ the wave function evolving under the free Hamiltonian (i.e. $h_f = -\frac{1}{2a^3}\frac{\partial^2}{\partial\Phi^2}$), the functions $\langle\Psi_{\omega}^{f} \mid \Psi\rangle$ can be analytically continued into a region of the *lower (upper)* complex half-plane. The following step consists in finding the poles of the functions $\langle\Psi_{\omega} \mid \Psi\rangle$ where $\{|\Psi\rangle\}$ is the basis of the complete Hamiltonian (the analytical domains must be large enough to contain these poles, which turn out to be the resolvent poles). Precisely, the following analytical continuation for $z \in C_{+}$ and $z' \in C_{-}$ are defined:

$$(\rho|\Psi_{z}, \Psi_{z'}) = cont_{\omega\to z} cont_{\omega'\to z'} (\rho|\Psi_{\omega}, \Psi_{\omega'}), \tag{30}$$

$$(\Psi_{z}, \Psi_{z'}|A) = cont_{\omega\to z} cont_{\omega'\to z'} (\Psi_{\omega}, \Psi_{\omega'}|A) \tag{31}$$

where $cont_{\omega\to z}$ is an analytical continuation from ω to z, $(\rho|\Psi_{z}, \Psi_{z'})$ is considered analytic, and $(\Psi_{z}, \Psi_{z'}|A)$ has poles (for simplicity we will just assume two simple poles at $z = z_o^*$ and $z' = z_o$, see [32], (68) and (69) and [24]) for the physical origin of the poles. Then, using the Cauchy theorem and the residues of the analytical functions around the poles, it can be proved that, for any $A \in V_{VH}^{+}$ and $\rho \in V_{VH}^{R+'}$ the weak evolution equation for $\rho(t)$ reads:

$$\begin{aligned}(\rho(t)|A) = {} & \int_0^\infty d\omega(\rho_o|\Psi_\omega)(\Psi_{\omega'}|A) + e^{i(z_o^*-z_o)t}(\rho_o|\phi_{oo})(\phi_{oo}^*|A) \\ & + \int_\Gamma dz'(\rho_o|\phi_{0z'})(\tilde{\phi}_{0z'}|A)e^{i(z_o^*-z')t} + \int_{\Gamma'} dz(\rho_o|\phi_{z0})(\tilde{\phi}_{z0}|A)e^{i(z-z_o)t} \\ & + \int_{\Gamma^*} dz \int_\Gamma dz'(\rho_o|\phi_{zz'})(\tilde{\phi}_{zz}|A)e^{i(z-z')t}\end{aligned} \tag{32}$$

where $\rho_o = \rho_*$, $z_o(z_o^*)$ is the pole (that we have considered unique for simplicity, see [32] (70)), $\Gamma(\Gamma^*)$ is a curve in the lower (upper) half-plane containing the corresponding pole, and

$$(\rho_o|\phi_{00}) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} (\rho_o|\Psi_\omega, \Psi_{\omega'}), \tag{33}$$

$$(\tilde{\phi}_{00}|A) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} 4\pi^2(\omega - z_o^*)(\omega' - z_o)(\Psi_\omega, \Psi_{\omega'}|A), \tag{34}$$

$$(\rho_0|\phi_{0z'}) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} (\rho_o|\Psi_\omega, \Psi_{\omega'}), \tag{35}$$

$$(\tilde{\phi}_{z0}|A) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} 2\pi i(\omega - z_o^*)(\Psi_\omega, \Psi_{\omega'}|A), \tag{36}$$

$$(\rho_o|\phi_{z0}) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} (\rho_o|\Psi_\omega, \Psi_{\omega'}), \tag{37}$$

$$(\tilde{\phi}_{0z'}|A) = -cont_{\omega\to z_o^*} cont_{\omega'\to z_o} 2\pi i(\omega' - z_o^*)(\Psi_\omega, \Psi_{\omega'}|A), \tag{38}$$

$$(\rho_o|\phi_{zz'}) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} (\rho_o|\Psi_\omega, \Psi_{\omega'}), \tag{39}$$

$$(\tilde{\phi}_{zz'}|A) = cont_{\omega\to z_o^*} cont_{\omega'\to z_o} (\Psi_\omega, \Psi_{\omega'}|A) \tag{40}$$

Therefore, $\{|\Psi_\omega), |\phi_{00}), |\phi_{z0}), |\phi_{0z'}), |\phi_{zz'})\}$ is a basis obtained by analytical continuation, $\{(\Psi_\omega|, (\phi_{00}|, (\phi_{z0}|, (\phi_{0z'}|, (\phi_{zz'})|\}$ is the corresponding cobasis and (see [32], (73)) and:

$$\begin{aligned} &(\Psi_\omega|\Psi_{\omega'}) = \delta(\omega - \omega') \quad (\tilde{\phi}_{00}|\phi_{00}) = 1, \\ &(\Psi_\omega|\phi_{00}) = (\tilde{\phi}_{00}|\Psi_{\omega'}) = 0 \end{aligned} \tag{41}$$

If, $z_o = \tilde{\omega}' - i\frac{\gamma}{2}$ and $\gamma > 0$ then $z_o^* - z_o = i\gamma$ and the five terms of eq. (32) can be interpreted as follows:

- (i) The first term is the decoherence-equilibrium term contained in V_{VH}^S, which is constant in time and equal to $(\rho_*|A)$.
- (ii) The second term contains the factor $e^{-\gamma t}$ and, therefore, it is the exponential *decaying* term. We will call this term $e^{-\gamma t}(\rho_1|A)$
- (iii) The sum of the remaining three terms is known as "background term" which is relevant only at the beginning (Zeno effect) and at the end (Khalfin effect) of the evolution (see e.g., [24]). Therefore, they can be neglected in the central long exponential period. Moreover, when $t \to \infty$, it vanishes as a linear sum of inverse powers of time.

2.2.3 *The Exponential Reduction (Reduction III)*

Summing up, the exponential decay, we were seeking for, is described in point (ii). If we only want to retain this exponential behavior by neglecting the background term, we have to introduce a further reduction, which we call *reduction III* or *exponential reduction*. Let us define a reduced state as

$$(\rho_r| = (\rho|\Pi_+ \tag{42}$$

where the projector Π_+ is defined as

$$\Pi_+ = \left(|\phi_{00}) + \int_0^\infty |\Psi_\omega)d\omega\right)\left((\tilde{\phi}_{00}| + \int_0^\infty (\Psi_\omega|d\omega\right) \tag{43}$$

From (41) we see that Π_+ is in fact a projector. Then, $(\rho_r(t)|$ results

$$(\rho_r(t)| = (\rho(t)|\Pi_+ = \int_0^\infty (\rho|\Psi_\omega)(\Psi_\omega|d\omega + (\rho|\phi_{00})(\tilde{\phi}_{00}|e^{i(z_o^*-z_o)t} = (\rho_*| + (\rho_1(t)| \tag{44}$$

and it only contains a constant and a decaying terms. Therefore, by means of the three reductions (van Hove, analytical and exponential) embodied in projector Π_+ , we have obtained the exponentially decaying evolution towards the decohered equilibrium state. Up to this point, we have used the spaces defined in (29). However, we could also have chosen the spaces

$$V_{VH}^{-}=V_{VH}^{S}\oplus V_{VH}^{R-}\qquad V_{VH}^{-'}=V_{VH}^{S'}\oplus V_{VH}^{R-'},\tag{45}$$

$$V_{VH}^{R-}=\Phi_{-}\otimes\Phi_{-}\qquad V_{VH}^{R-'}=\Phi_{+}\otimes\Phi_{+}\tag{46}$$

In this case, we would have obtained the projector Π_- and the factor $e^{\gamma t}$ (with $\gamma>0$), which does not represent an exponentially decaying term, but an exponentially *growing* term, corresponding to an evolution that leads to equilibrium in the past. This means that the *time-reversal invariant laws* of quantum mechanics give rise to a pair of *time-symmetric twins*: one including the factor $e^{-\gamma t}$, that describes the decaying of an unstable quantum state *towards equilibrium* (a *dissipative-like* process), the other including the factor $e^{\gamma t}$, that describes the growing of an unstable quantum state *from equilibrium* (an *antidissipative-like* process). Then we have a couple of time-symmetric twin; but of course, the global grounds of papers [2] and [31], we must just choose the first expanding twins as it is explained in these references.

3 Conditional Entropy of the Universe

In this section we will define a "conditional entropy" which is related to the phenomenological entropy of thermodynamics as we will see in the next section. Let us define a quantum conditional entropy:

$$S_c=-\left\langle\rho_r(t)\log[\rho_r(t)\rho_*^{-1}]\right\rangle\tag{47}$$

where $\rho_r(t)$ is the density operator, ρ_* is the equilibrium density operator, and $\langle\ldots\rangle$ symbolizes the trace. This entropy becomes the quantum Gibbs entropy in the particular case $\rho_*=\text{const}$. Therefore, the problem of irreversibility consists in finding a growing entropy that evolves towards a final equilibrium value. In order to face this problem, let us write (44) with the explicit time dependence of each term:

$$\rho_r(t)=\rho_*+e^{-\gamma t}\rho_1\tag{48}$$

where ρ_* and ρ_1 are constant operators. This is a "weak" quantum equation, since it is only valid for the state $\rho_r(t)$, completely reduced under the action of the projector Π_+. In this sense, $\rho_r(t)$ can be conceived as a particular coarse-grained state. Developing (47) when we replace $\rho_r(t)$ from (48) we have

$$S_c=-\left\langle(\rho_*+e^{-\gamma t}\rho_1)(\log[\rho_*+e^{-\gamma t}\rho_1]-\log[\rho_*^{-1}]\right\rangle\tag{49}$$

Rewriting $\rho_*+e^{-\gamma t}\rho_1=\rho_*(1+e^{-\gamma t}\rho_1\rho_*^{-1})$ in the first logarithm we obtain:

$$S_c=-\left\langle(\rho_*+e^{-\gamma t}\rho_1)(\log[\rho_*]+\log[1+e^{-\gamma t}\rho_1\rho_*^{-1}]-\log[\rho_*^{-1}]\right\rangle\tag{50}$$

Expanding the second logarithm in Taylor polynomials and keeping up to the second order we get:

$$S_c=-\left\langle(\rho_*+e^{-\gamma t}\rho_1)e^{-\gamma t}\rho_1\rho_*^{-1}\right\rangle\tag{51}$$

and finally

$$S_c=-e^{-\gamma t}\left\langle\rho_*\rho_1\rho_*^{-1}\right\rangle-e^{-2\gamma t}\left\langle\rho_1^2\rho_*^{-1}\right\rangle\tag{52}$$

The order in the operators in the first term of the r.h.s can be interchanged since $\langle\rangle$ is a trace, thus $\langle\rho_*\rho_1\rho_*^{-1}\rangle = \langle\rho_1\rangle$, so

$$S_c = -e^{-\gamma t}\langle\rho_1\rangle - e^{-2\gamma t}\left\langle\rho_1^2\rho_*^{-1}\right\rangle \tag{53}$$

On the other hand $\langle\rho(t)\rangle = \langle\rho_*(t)\rangle + \langle\rho_1(t)\rangle e^{-\gamma t}$. Because the wave function is normalized, then $\langle\rho(t)\rangle = \langle\rho_*(t)\rangle = 1$, then from equation (48) $\langle\rho_1(t)\rangle = 0$ so:

$$S_c = -e^{-2\gamma t}\langle\rho_1^2\rho_*^{-1}\rangle = e^{-2\gamma t}\int_\Gamma \rho_1^2\rho_*^{-1}d\Gamma + O\left(\frac{\hbar}{S}\right) \tag{54}$$

where the symbols in the r.h.s. are classical states, obtained by the Wigner transformation (see [19], [13], and [41] for details) and S is the action. Also (48) can be transformed via the Wigner transformation [41]. The magnitude $S_c(t)$ from (54) would be our candidate for "conditional entropy", that is, an entropy derived from fundamental equations by reduction (see [16] for further details). In fact, $S_c(t)$ increases with time and, for $t\to\infty$, $S_c(t)\to 0$ and $\rho(t)\to\rho_*$. Moreover, if σ is the entropy production,

$$\sigma = \dot{S} > 0, \tag{55}$$

$$\dot{\sigma} = \ddot{S} < 0 \tag{56}$$

This means that our fundamental entropy $S_c(t)$ satisfies all the necessary requirements for being an irreversible growing entropy that reaches its maximum value at equilibrium [58]. We have proved that a thermodynamic arrow of time associated to this entropy exists and it is based on the cosmological framework which is consistent with our scheme of the arrow of time.

4 The Gibbs and Thermodynamics Entropies

To complete the panorama in this section we deduce the relation among the conditional entropy S_c with the Gibbs entropy S_G and the thermodynamical entropy. We can write (47) as

$$S_c = -\langle\rho_r(t)\log[\rho_r(t)]\rangle + \langle\rho_r(t)\log[\rho_*(t)]\rangle \tag{57}$$

If we perform a differentiation in both terms

$$-dS_c + d\langle\rho_r(t)\log[\rho_*(t)]\rangle = d\langle\rho_r(t)\log[\rho_r(t)]\rangle \tag{58}$$

As $\rho_*(t)$ is the equilibrium state, it can be written as $\rho_*(t) = Z^{-1}e^{-\beta E}$ where $\beta = \frac{1}{T}$, then

$$\log[\rho_*(t)] = \log(Z^{-1}) - \beta E \tag{59}$$

Because $Z = \text{const}$ then

$$d\langle\rho_r(t)\log[\rho_*(t)]\rangle = -\beta d\langle\rho_r(t)E\rangle \tag{60}$$

where $\langle\rho_r(t)E\rangle$ is the total energy of the system, which in the case of a generic open system (not in the case of the universe which is a closed system) would be the increment of energy or heat dQ interchanged between the system and the environment. Thus from (58)

$$-d\langle\rho_r(t)\log[\rho_r(t)]\rangle = \frac{1}{T}dQ + dS_c \tag{61}$$

If we call $-S_G(\rho_r(t)) = \langle \rho_r(t)\log[\rho_r(t)]\rangle$ the usual Gibbs entropy, (61) is

$$dS_G = \frac{1}{T}dQ + dS_c \tag{62}$$

Therefore

(i) Equation (62) is the increment of ordinary Gibbs entropy in a period of time dt.
(ii) The first term of (62) of the l.h.s., is $\frac{dQ}{T}$, where dQ is the heat interchanged between the system and the environment. If the system is near equilibrium we have $dS_c \cong 0$ and we obtain the usual definition of thermodynamic entropy $dS_G = dS = \frac{dQ}{T}$.
(iii) The second term of (62) is the conditional entropy production in the period of time dt according to (55) and (56).

Going back to the case of the universe (a closed system) we have $dQ = 0$ and

$$dS_G = dS_c \tag{63}$$

so S_G satisfies (55) and (56). All these facts completes our demonstration.

5 Conclusions

In this work we have shown that it is possible to introduce a Gibbs entropy in a cosmological framework in the early stages of the universe. Using a Wheeler-De Witt model, using self induced decoherence formalism, we have found that the decohered states reaches the equilibrium with a damped factor related with the interaction potential of the universe. The irreversibility of this process implies that another example of the thermodynamic arrow of time has being founded based on a cosmological framework.

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Appendix A: The Self Induced Approach to Decoherence

A.1 Introduction

The SID approach relies on the general idea that the interplay between observables and states is a fundamental element of quantum mechanics [50]. Formally the mathematically rigorous departing point consists in the choice of an algebra of operators A as the primitive element of the theory: the observables are the self-adjoint operators of A. In the original formulation of the algebraic formalism, the algebra of observables is a C*-algebra. The GNS theorem (Gel'fand-Naimark-Segal) proves that the traditional Hilbert space formalism is a particular representation of this algebraic formalism; the algebra of observables is thereby given by concrete representation as a set of self-adjoint bounded operators on a separable Hilbert space. Nevertheless, since the C*-algebraic framework does not admit unbounded operators, it is necessary to move to a less restrictive framework in order to adequate this kind of operators. The self-induced approach adopts a nuclear algebra [60] as the algebra of observables A: its elements are nuclei or kernels, that is, two variables distributions that can be thought of as generalized matrices [28]. By means of a generalized version of the

GNS theorem [42, 43], it can be proved that this nuclear formalism has a representation in a rigged Hilbert space: the appropriate rigging provides a mathematical rigorous foundation to unbounded operators [6]. In fact, the nuclear spectral theorem of Gel'fand and Maurin establishes that, under very general mathematical hypotheses (quite reasonable from a physical point of view), for every CSCO (complete set of commuting observables) of essentially self-adjoint unbounded operators, there is a rigged Hilbert space where such a CSCO can be given a generalized eigenvalue decomposition, meaning that a continuum of generalized eigenvalues and eigenvectors may thereby be associated with it. In order to find the appropriate rigging, the nuclear algebra is used to generate two additional topologies: one of them corresponds to a nuclear space, which is the space of generalized observables V_O; the other corresponds to the dual of the space V_O, and it is the space V_S of states.

A.2 The Formalism

After this (non-essential) mathematical introduction we will present the SID formalism. Following [3, 50, 51] we will symbolize an observable belonging to V_O by a round ket $|O)$ and a state belonging to V_S by a round bra $(\rho|$. The result of the action of the round bra $(\rho|$ the round ket $|O)$ is the mean value of the observable $|O)$ in the state $(\rho|$:

$$\langle O\rangle_\rho = (\rho|O) \tag{64}$$

If the basis is discrete, $\langle O\rangle_\rho$ can be computed as usual, that is, as $Tr(\rho O)$. But if the basis is continuous, $Tr(\rho O)$ is not well defined; nevertheless, $(\rho|O)$ can always be rigorously defined since $(\rho|$ is a linear functional belonging to V_S acting onto an operator $|O)$ belonging to V_O. In order to see how decoherence works from the new approach, let us consider the simplest case, that is, a quantum system whose Hamiltonian has a continuous spectrum $\omega \in [0, \infty)$:

$$H|\omega\rangle = \omega|\omega\rangle \quad \omega \in [0, \infty) \tag{65}$$

where ω and $|\omega\rangle$ are the generalized eigenvalues and eigenvectors of H respectively. In the simplest model, the CSCO of this system is just $\{H\}$. A generic observable $|O)$ can be expressed in terms of the eigenbasis $\{|\omega\rangle\langle\omega'|\}$ as:

$$|O) = \int\int \hat{O}(\omega,\omega')|\omega\rangle\langle\omega'|d\omega d\omega' = \int\int \hat{O}(\omega,\omega')|\omega,\omega')d\omega d\omega' \tag{66}$$

where $|\omega,\omega') = |\omega\rangle\langle\omega'|$ and $\hat{O}(\omega,\omega')$ are the coordinates of the kernel $|O)$. The Hamiltonian in the eigenbasis $\{|\omega,\omega')\}$ reads:

$$|H) = \int \omega|\omega\rangle\langle\omega|d\omega = \int\int \omega\delta(\omega-\omega')|\omega,\omega')d\omega d\omega' \tag{67}$$

Then, $\omega\delta(\omega-\omega')$ must be one of the $\hat{O}(\omega,\omega')$, since H is one of the observables belonging to V_O. Moreover, all the observables commuting with H and sharing the eigenbasis $\{|\omega,\omega')\}$ must be:

$$|O) = \int O(\omega)|\omega\rangle\langle\omega|d\omega = \int\int O(\omega)\delta(\omega-\omega')|\omega,\omega')d\omega d\omega' \tag{68}$$

where now $O(\omega)$ supplies the values of all the components of $|O)$ in the basis $\{|\omega;\omega')\}$. Therefore, $O(\omega)\delta(\omega-\omega')$ must be one of the $\hat{O}(\omega,\omega')$. But, of course, we also need observables which do not commute with H and whose $\hat{O}(\omega, \omega')$ are different than $O(\omega)\delta(\omega - \omega')$; then, with no loss of physical generality [20] we can say that, in the general case:

$$\hat{O}(\omega, \omega') = O(\omega)\delta(\omega - \omega') + O(\omega, \omega') \tag{69}$$

where $O(\omega, \omega')$ is a regular function whose precise mathematical properties are listed in [18]. Therefore, a generic observable $|O)$ reads (see [61]):

$$|O) = \int O(\omega)|\omega)d\omega + \int\int O(\omega, \omega')|\omega, \omega')d\omega d\omega' \tag{70}$$

where $|\omega) = |\omega\rangle\langle\omega|$ and $|\omega, \omega') = |\omega\rangle\langle\omega'|$ are the generalized eigenvectors of the observable $|O)$. We will call the first term of the r.h.s of (70) O_S (the singular part of the observable $|O)$), and the second term of the r.h.s of (70) O_R (the regular part of the observable $|O)$). The observables $|O)$ of the form (70) define what we will call "*Van Hove space*", $V_O^{VH} \subset V_O$; whose basis is $\{|\omega), |\omega; \omega')\}$. On the other hand, states are represented by linear functionals belonging to a space V_S^{VH}, which is the dual of V_O^{VH}. Therefore, a generic state $(\rho|$ can be expressed as:

$$(\rho| = \int \rho(\omega)(\omega|d\omega + \int\int \rho(\omega, \omega')(\omega, \omega'|d\omega d\omega' \tag{71}$$

where $\rho(\omega, \omega')$ is a regular function, and $\rho(\omega)$ and $\rho(\omega, \omega')$ satisfy the properties $\rho \geq 0$, $(\rho|I) = 1$ (where $|I)$ is the identity operator) and those listed in [18]. $\{(\omega|, (\omega, \omega'|\}$, the basis of V_S^{VH}, i.e. the cobasis of $\{|\omega), |\omega, \omega')\}$, is defined by the following relations:

$$(\omega|\omega') = \delta(\omega - \omega') \qquad (\omega, \omega''|\omega', \omega''') = \delta(\omega - \omega'')\delta(\omega' - \omega''') \qquad (\omega|\omega', \omega'') = 0 \tag{72}$$

Given the expressions (70) and (71) for $|O)$ and $(\rho|$ respectively, decoherence follows in a straightforward way. According to the unitary von Neumann equation, the evolution of $(\rho|$ is given by:

$$(\rho(t)| = \int \rho(\omega)(\omega|d\omega + \int\int \rho(\omega, \omega')e^{-i(\omega-\omega')t}(\omega, \omega'|d\omega d\omega' \tag{73}$$

Therefore, the mean value of the observable $|O)$. in the state $(\rho(t)|$ reads:

$$\langle O\rangle_{\rho(t)} = (\rho(t)|O) = \int \rho^*(\omega)O(\omega)d\omega + \int\int \rho^*(\omega, \omega')e^{-i(\omega-\omega')t}O(\omega, \omega')d\omega d\omega' \tag{74}$$

Since $\rho(\omega, \omega')$ and $O(\omega, \omega')$ are regular functions, it is natural to require that $\rho(\omega, \omega')O(\omega, \omega')$ would be L_1 in the variables $\omega - \omega'$ (see ([51]) for details). Then when we take the limit for $t \to \infty$, we can apply the Riemann-Lebesgue theorem according to which the second term of the right hand side of the last equation vanishes. Therefore:

$$\lim_{t\to\infty} \langle O\rangle_{\rho(t)} = \lim_{t\to\infty} (\rho(t)|O) = \int \rho(\omega)O(\omega)d\omega \tag{75}$$

But this integral is equivalent to the expectation value of the observable O in a new state $(\rho_*|$:

$$(\rho_*| = \int \rho(\omega)(\omega|d\omega \tag{76}$$

where the off-diagonal terms have vanished. Therefore, we obtain the limit:

$$\lim_{t\to\infty}\langle O\rangle_{\rho(t)}=\langle O\rangle_{\rho_*} \tag{77}$$

or

$$W-\lim_{t\to\infty}\rho(t)=\rho_*$$

To complete the subject it would be necessary to obtain the classical limit of the system. But really this limit is not relevant for the subject of this paper so we refer the reader to papers [24] to [14].

As this presentation shows, decoherence does not require the interaction of the system of interest with the environment: *a single closed quantum system can decohere* (in the case of this paper the whole universe). The diagonalization of the density operator does not depend on the openness of the system, but on the continuous spectrum of the Hamiltonian system. This means that the problem of providing a general criterion for discriminating between system and environment vanishes in the self-induced approach. This fact leads to an additional advantage of the new way of conceiving decoherence. As we have seen, in many cases the einselection approach in EID requires to introduce assumptions about the observables which will classically behave in order to decide where to place the boundary between system and environment. The new approach, on the contrary, provides a mathematically precise definition of the observables such that the system observed by these operators decoheres, i.e. the Van Hove observables of space V_O^{VH}.

A.3 Decoherence Time

As a consequence of the Riemann-Lebesgue theorem, full decoherence strictly occurs when $t\to\infty$. However, as in any exponential decaying process, there is a characteristic decaying time that can be considered as the time at which, in practice, the decaying is approximately completed. But in a closed system, the relaxation time and the decoherence time are the same, since complete decoherence (i.e. relaxation in this case) takes place at the same time than equilibrium (see [22]). In the next sections we will compute the relaxation time of a self-induced decoherence process as the characteristic decaying time of the fluctuating term in the expression of $\langle O\rangle_{\rho(t)}$ of (74). In order to study and compute the relaxation time, we will use the standard theory of analytical continuation in the scattering quantum theory (see, e.g., [4, 8]) and its extension to the Liouville-von Neumann space (see, e.g., [17, 32]). By means of this theory, we can compute the decoherence time in terms of the poles corresponding to the functions involved in the fluctuating term of $\langle O\rangle_{\rho(t)}$ (see (74)). Let

$$(\rho_R(t)|O_R)=\int_0^{+\infty}\int_0^{+\infty}\rho(\omega,\omega')O(\omega,\omega')e^{-i(\omega-\omega')t}d\omega d\omega' \tag{78}$$

where $|O_R)$ and $|\rho_R)$ are the regular parts of $|O)$ and $|\rho)$ respectively. In this equation, we can introduce the following change of variables:

$$\lambda=\frac{1}{2}(\omega+\omega')\quad \nu=\omega-\omega'\quad d\omega d\omega'=Jd\lambda d\nu=d\lambda d\nu \tag{79}$$

Then,

$$(\rho_R(t)|O_R)=\int_0^{+\infty}d\lambda\int_{-2\lambda}^{2\lambda}\rho'(\nu,\lambda)O'(\nu,\lambda)e^{-i\nu t}d\nu \tag{80}$$

where $\rho'(\nu,\lambda)=\rho(\omega,\omega')$, $O'(\nu,\lambda)=O(\omega,\omega')$ and the new limits of the integrals are due to the fact that $\omega,\omega'\geq 0$. Now we promote the real variable ν to a complex variable Z; if the function $\rho'(Z,\lambda)O'(Z,\lambda)$ has no poles in the upper Z-half-plane, we obtain

$$(\rho_R(t)|O_R)=\int_0^{+\infty}d\lambda\int_{C(-2\lambda,2\lambda)}\rho'(Z,\lambda)O'(Z,\lambda)e^{-iZt}dz \tag{81}$$

where $C(-2\lambda,2\lambda)$ is any curve that goes from -2λ to 2λ by the upper complex half-plane. If the function $\rho'(Z,\lambda)O'(Z,\lambda)$ has, let us say, a pole at $Z_0=\tilde{\omega}+i\gamma$ in the upper half-plane, we can, as usual, decompose $C(-2\lambda,2\lambda)=\Gamma(-2\lambda,2\lambda)\cup C_{Z_0}$ where C_{Z_0} is a residue-circle around the pole Z_0 and $\Gamma(-2\lambda,2\lambda)$ is the remaining 'background' curve. If, as usual, we neglect the background, only the factor $e^{i\frac{Z}{\hbar}t}$ becomes relevant at the pole Z_0, this factor reads

$$e^{i\frac{Z_0}{\hbar}t}=e^{i\frac{\tilde{\omega}+i\gamma}{\hbar}t}=e^{i\frac{\tilde{\omega}}{\hbar}t}e^{-\frac{\gamma}{\hbar}t} \tag{82}$$

where $e^{-\frac{\gamma}{\hbar}t}$ is a dumping factor appearing in the regular fluctuating term of $\langle O\rangle_{\rho(t)}=(\rho(t)|O)$. Therefore, the decoherence time can be computed as the characteristic decaying time of the process as

$$t_D=\frac{\hbar}{\gamma} \tag{83}$$

In the first approximation,the dumping factor γ in equation (83) is proportional to the interaction Hamiltonian. With this fact in mind and in the particular cases like e.g. the Friedrich model studied in papers [17] and [32] we obtain

$$t_D=\frac{\hbar}{2\pi|V_\Omega|^2} \tag{84}$$

being $H_{\text{int}}\approx 2\pi|V_\Omega|^2$ the interaction function. It is clear that, if the interaction vanishes $t_D\to\infty$. In turn, if the characteristic energy $2\pi|V_\Omega|^2\sim V$ is, let us say, 1 electron-volt (a natural energy for quantum atomic interactions, see e.g. [49]), the decoherence time is $\sim 10^{-15}$ s.

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